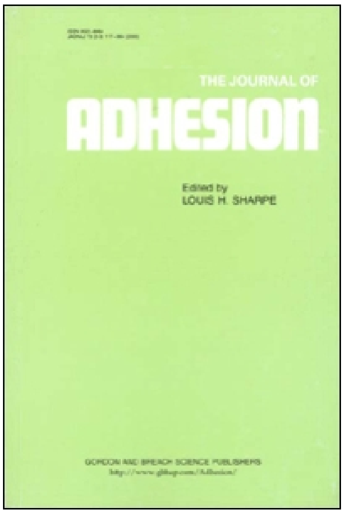

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A Comparison of Linear with Nonlinear Viscoelastic Solutions for Shear Stress Concentration in Double Lap Joints

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The correspondence principle based on the Maxwell model and a nonlinear viscoelastic solution involving an iterative scheme are used to describe the time dependent variation of the adhesive maximum shear stress in adhesively bonded double lap joints. The results indicate that if the correspondence principle is applied, the use of Maxwell chain is necessary to approximate the continuous change in the relaxation time and to coincide with the results calculated using the nonlinear viscoelastic theory.

KEY WORDS Double lap joint; linear viscoelastic analysis; correspondence principle; nonlinear viscoelastic analysis; adhesive maximum shear stress; relaxation time.

INTRODUCTION

Elastic analysis of double lap joints reveals that the adhesive shear stresses vary along the overlap length (L) according to the relation[1,2]

$$\tau = -(P/4A)\cosh(x/A)/\sinh(L/2A) \qquad (1)$$

where,

$$A = (\delta dE/2G)^{1/2}, \qquad (2)$$

E and G are the substrate (adherend) Young's modulus and adhesive shear modulus, δ and d are adhesive and adherend thicknesses respectively, and P is the axial load in the main plate (Fig. 1).

It should be noted that a complete linear-elastic analysis is expected to result in corner singularities at the locations where the adhesive-adherend interface intersects the free edge.[2] Due to the elastoplastic or viscoelastic-plastic nature of the adhesive material, however, such high levels of stresses are redistributed to levels encountered in usual engineering stress concentration cases.[2,3,4] Peel stresses are also expected to be created at the overlap edges of a double lap joint by moment forces compensating the absence of shear forces at the free ends of the (very thin)

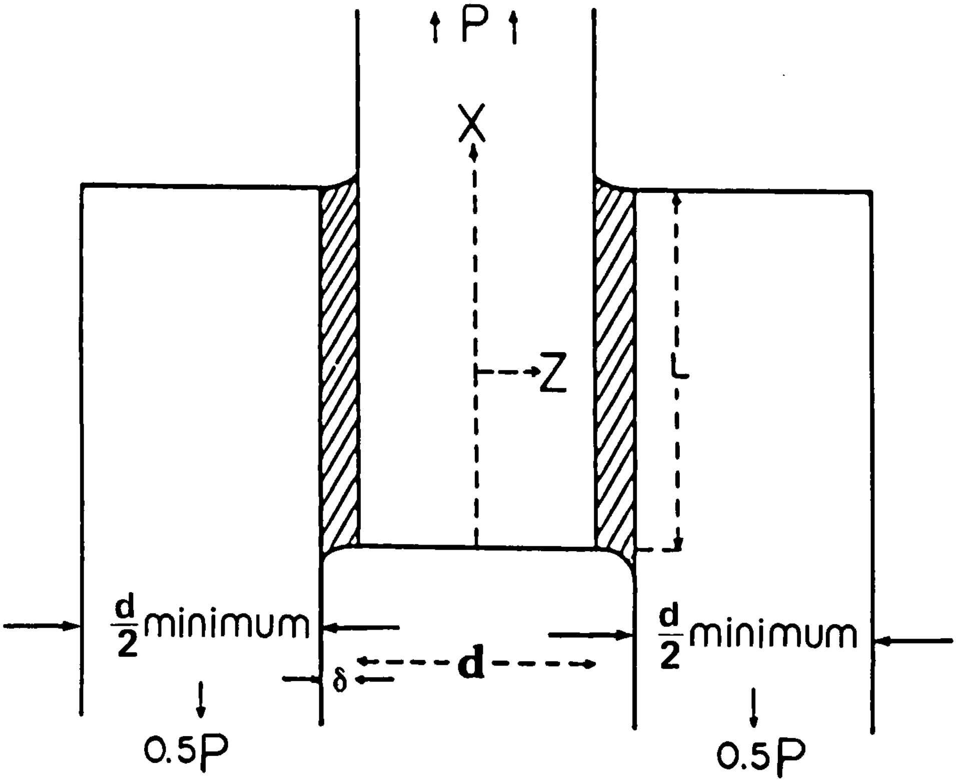

FIGURE 1 Double Lap Joint Geometry.

adhesive layers. This paper, however, focuses on shear stress concentration at the overlap edges and will not analyse peel stresses, as peel deformations are expected to be small due to the symmetric nature of the double lap specimen and also due to the assumption of rigid adherends.

Equation (1) maximizes the adhesive shear stress at the overlap edges where $x = \pm L/2$. Apparently, only one needs to consider this maximum level of the shear stress for design purposes. The time dependent variation of the maximum stress is of critical importance to the designer as most adhesives are viscoelastic polymers.[5]

For nonlinear viscoelastic analysis the most widely used constitutive model involves a "power-law" compliance[5]

$$D(t,\tau) = D_0 + D_1\,(t/e^{-\theta\tau})^n. \tag{3}$$

In Equation (3) D_0 is the instantaneous creep compliance, and D_1 (transient creep compliance), n (power factor for time) and, θ (strength of stress contribution in time shift) are material parameters which represent the nonlinear and time dependent material behavior. Typical values for epoxies[2] which will be used in this paper are $D_1 = 1 \times 10^{-7}$ (psi-sec^n)$^{-1}$, $\theta = 1.25 \times 10^{-4}$ psi^{-1}, and $n = 0.2$. The instantaneous compliance can be assumed to be equal to the inverse of the elastic modulus for all practical purposes. Note that Equation (3) incorporates a "reduced time" (t/a_τ) where the time reduction is accomplished by a stress-dependent shift factor,

$$a_\tau = \exp(-\theta\tau). \tag{4}$$

This paper will present solutions describing the time variation of the maximum adhesive shear stress in double lap joints using the linear correspondence principle, which provides a closed-form solution for the maximum adhesive shear stress, and also using a nonlinear compliance method which provides numerical results for the nonlinear partial differential equation governing the state of shear stress. The nonlinear solution involves a quasi-elastic method utilizing iteration by successive differentiation as described by Weitsman.[2] Adhesive maximum shear stress values will be evaluated using both methods and the results of linear analysis will be matched to those of nonlinear analysis using an additional numerical procedure. This second numerical procedure involves the use of variable relaxation time which is changed as a function of time. The paper, therefore, illustrates the capability of obtaining accurate solutions using the correspondence principle if variable relaxation time is used.

The general form of the correspondence principle[6,7] states that if the solution of an elastic problem is known, the Laplace transform of the solution to the corresponding linear viscoelastic problem may be found by replacing the elastic constants with their linear viscoelastic counterparts, and the actual loads by their Laplace transforms. This modified transform is inverted subsequently to obtain the solution in the time domain. It should be noted that for some cases Laplace transformation is not possible. Such cases occur when regions over which boundary conditions act are changing in the course of time. The case study presented in this paper which involves the double lap joint problem (Fig. 1), however, does not fall into this category.

ANALYTICAL CONSIDERATIONS

Simplification of the Elastic Solution

For common overlap geometries and adhesive-adherend materials, the hyperbolic part, $\cosh(x/A)/\sinh(L/2A)$, of Equation (1) can be assumed to be equal to one at the overlap ends $(x = L/2)$ and their close vicinity. In fact, for the material parameters: substrate modulus $E_s = 10 \times 10^6$ psi, adhesive shear modulus, $G_a = 5 \times 10^5$ psi and the geometrical parameters: adhesive thickness, $\delta = 3 \times 10^{-3}$ in, adherend thickness, $d = 0.5$ in, to be used in this paper, we get $\cosh(x/A)/\sinh(L/2A) = 1.0005690$ for the adhesive layer location $x = L/2 = 1/2$ at the overlap edges. Note that this hyperbolic term progressively deviates from unity for the following conditions: shorter overlap lengths, L, larger values of the product $\delta d E_s$, and smaller values of the adhesive modulus, G_a. With this assumption, simple substitution results in the ratio of the maximum stress, τ_{max}, to the average stress, τ_{avg} in the form[8]

$$\tau_{max}/\tau_{avg} = [L/(2\delta dE)^{1/2}](G)^{1/2} \tag{5}$$

where the average shear stress (per unit width) in the adhesive is given by

$$\tau_{avg} = -P/2wL \qquad (6)$$

with w = adherend width = 1 for the present solution. The assumption of unity width for the adherends is consistent with the elastic plane strain solution (*i.e.* strain, $\epsilon = 0$ in the width, w, direction) utilized in this paper. It should be noted that the use of the plane strain condition for lap geometries is a well-accepted procedure routinely utilized by other researchers.[2,9] The simple form of Equation (5) allows the application of the correspondence principle to account for the time effects.

Linear Viscoelastic Solution

Based on the correspondence principle, the general stress problem is the same for elastic and linear viscoelastic structures in the sense that three basic sets of equations must be satisfied. These equations represent equilibrium conditions, kinematic relations and constitutive equations. The only difference between the two types of problems is that for viscoelastic structures Hooke's law is replaced by a linear viscoelastic constitutive equation of the form[10]

$$\ldots \approx{B}_{ijkr}\ddot{\sigma}_{kr} + \tilde{B}_{ijkr}\dot{\sigma}_{kr} + \sigma_{ij} = C_{ij} + C_{ijkr}L_{kr} + \tilde{C}_{ijkr}\dot{L}_{kr} + \ldots \qquad (7)$$

where dots indicate time derivatives, L_{ij} is the Lagrangian linear strain tensor and C_{ij}, C_{ijkr} . . . , $\tilde{B}_{ijkr}$ . . . are coefficients. When the coefficients of Equation (7) are assumed to be constants and the derivatives of the displacements are small, then Equation (7) describes a linear viscoelastic material. Linear combinations of spring and dashpot models can be used to describe Equation (7); for example a series combination of a linear spring representing elastic modulus, E, and a dashpot representing viscosity coefficient, μ, describes the Maxwell body (Fig. 2) with the constitutive equation

$$\tilde{B}_{ijkr}\dot{\sigma}_{kr} + \sigma_{ij} = \tilde{C}_{ijkr}\dot{L}_{kr}. \qquad (8)$$

The one dimensional form of Equation (8) is

$$(\mu/E)\dot{\sigma} + \sigma = \mu\dot{\epsilon}. \qquad (9)$$

Overstress versions of the Maxwell model have been used to describe the constitutive behavior of structural adhesives.[11,12,13] Such models include sliding elements in parallel with the dashpot and in series with the spring to represent overstress conditions: elastic limit stress and ultimate (or yield) stress, respectively, which mark constitutive behavior changes in the adhesive material through some failure mechanism.

When one takes the Laplace transform of both sides of Equation (8), a linear stress-strain relation is obtained in the transform domain. In one dimension, such a relation can be expressed as

$$\bar{\sigma} = \bar{E}^*(s)\bar{\epsilon}\,, \qquad (10)$$

where the exact form of the coefficient $\bar{E}^*(s)$ depends on the mechanical spring—

dashpot model chosen; for example, for a Maxwell model consisting of a spring "E" and dashpot "μ":

$$\bar{E}^*(s) = (sE)/[s + (E/\mu)]. \tag{11}$$

Shear stresses (τ) and strains (γ) can be related in the transform domain in a manner similar to Equation (10):

$$\bar{\tau} = \bar{G}^*(s)\bar{\gamma}\,. \tag{12}$$

Based on Maxwell model; the coefficient $\bar{G}^*(s)$ can be expressed as

$$\bar{G}^*(s) = (sG)/[s + (G/\mu_s)] \tag{13}$$

where G is the elastic shear modulus and μ_s is the coefficient of viscosity in shear which is related to the coefficient of viscosity in tension, μ,[14] by

$$\mu_s = (\mu)/[2(1 + \upsilon)] \tag{14}$$

where υ is the Poisson's ratio. Note that Equation (14) assumes equivalent relaxation times in shear and tension, *i.e.*

$$T = (\mu_s/G) = (\mu/E) \tag{15}$$

as suggested by Tobolsky.[15]

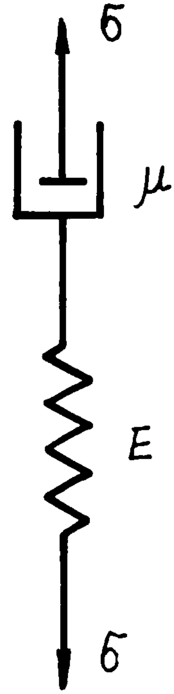

FIGURE 2 The Maxwell Model.

The correspondence principle can be applied to obtain the linear viscoelastic counterpart of Equation (5) in the following manner: If one assumes elastic adherends, the only elastic constant to be replaced in Equation (5) is G. In order to represent real structural adhesives and also to simplify the solution, the Maxwell model can be chosen to represent a viscoelastic constitutive equation. Therefore the $\overline{G}^*(s)$ coefficient which replaces G in the transform is defined by Equation (13).

The nonlinear viscoelastic solution mentioned earlier utilizes a creep compliance function (Equation 3) with four distinct material parameters. Such use of a creep compliance function facilitates displacement-type formulations based on shear-lag-type analysis. The nonlinear viscoelastic analysis presented in this paper, and applied earlier by Weitsman,[2] involves such treatment. With linear analysis, however, we can directly apply the condition of stress relaxation at the stress concentration location in double lap joints. Indeed, numerical results published earlier by Weitsman[2] were shown as space dependent shear stress curves plotted as functions of time and illustrated stress relaxation. The linear analysis presented here involves only two material parameters, G_a and μ_s which are combined in one explicit parameter, the relaxation time, T. It should be noted, however, that the eventual use of variable relaxation times also renders the originally linear method nonlinear as illustrated by Equation (23), which is obtained numerically.

Perhaps the most significant difference between the two methods presented is the fact that the nonlinear analysis is based on a creep compliance and, consequently, the adhesive strain must increase constantly for relaxation to take place (at the overlap edges) while the applied load level is fixed. With the linear method, however, the level of initially applied strain is kept constant while the stress, and the applied load are allowed to relax. With this method having a constant level of 2.04% shear strain (applied in present calculations) at the overlap edges does not mean that the creep process does not take place. In fact, continuous conversion of elastic strain (from the spring element) to plastic strain (in the dashpot element) tantamounts to a creep process except under fixed displacement conditions.

In order to obtain the closed form adhesive maximum shear stress expression based on the linear correspondence principle the relaxation condition is described mathematically by

$$\gamma(t) = \gamma_0 H(t) \tag{16}$$

where H(t) is the Heaviside unit step function and γ_0 is the level of applied constant shear strain. Using

$$\tau_{avg} = G\gamma_{avg} \tag{17}$$

and Equation (16), Equation (17) can be written in the transform domain as:

$$\tau_{max}(s) = [(L\gamma_0)/(2\delta dE)^{1/2}]\,(1/s)\,[\overline{G}^*(s)]^{3/2}. \tag{18}$$

Substitution of Equation (13) and inverse transformation results in:

$$\tau_{X=L/2}(t) = \{[L\gamma_0(G)^{3/2}]/(2\delta dE)^{1/2}\}\,\{e^{-(t/2T)}\,I_0(t/2T) - (t/T)\,e^{-(t/2T)}\,[I_0(t/2T) - I_1(t/2T)]\}. \tag{19}$$

where I_0 and I_1 refer to the modified Bessel functions of the first kind of orders 0 and 1 respectively.

Nonlinear Viscoelastic Solution Based on Power-Law Creep Compliance

In order to develop the nonlinear viscoelastic solution to the symmetrical double lap joint problem, Weitsman[2] first applies a variational formulation to the linear elastic problem which yields the solution given by Equation (1). The same method is then extended to the nonlinear viscoelastic case by using the nonlinear compliance given by Equation (3) to describe the shear strain behavior of the adhesive materials. As a result, the nonlinear partial differential equation

$$\tau''(x) = (1/\gamma'(\tau))\{[\tau(x)/\beta] - \gamma'(\tau)[\tau'(x)]^2\} \quad (20)$$

is obtained. This equation is solved by using a numerical iterative scheme which involves successive differentiation of Equation (20) along with Taylor expansions of differentials in terms of higher differentials of the previous sub-interval along the space coordinate. This numerical technique is similar to Picard's method for successive approximations.[16]

Iterative Method for Nonlinear Viscoelastic Solution

The iterative method used to solve Equation (20) is similar to the method used by Weitsman.[2] The first step in this method is successive differentiation of Equation (20) with respect to the space variable, x. The availability of first through fifth space derivatives of $\tau(t,x)$ is assumed to provide sufficient accuracy for the numerical method.

The next step involves expressing τ_i and $d\tau_i/dx$ by Taylor expansions with five terms assumed to provide sufficient accuracy, *i.e.*,

$$\tau_i = \tau_{i-1} + \Delta\tau'_{i-1} + \ldots + (1/4)\Delta_4\tau_{i-1}^{IV} \quad (21)$$

and

$$\tau'_i = \tau'_{i-1} + \Delta\tau''_{i-1} + \ldots + (1/4)\Delta_4\tau_{i-1}^{V} \quad (22)$$

The half overlap length $0 \leq x \leq (L/2)$ is divided into N equal sub-intervals of length $\Delta = L/2N$ with $N = 500$. For iteration purposes, each space iteration node is identified as $x_i = i\Delta (i = 1,2, \ldots ,N)$ and the corresponding shear stress to be calculated as $\tau(t,x_i) = \tau_i$.

The iteration process starts with initial guess values for τ'_0 to be substituted along with the boundary condition, $\tau(0) = \tau_0^{(1)} = 0$ into Equation (20) to compute τ''_0. τ'''_0, τ_0^{IV} and τ_0^{V} are then calculated using the derivatives of Equation (20).[2] These computed values can now be substituted into equations (21) and (22) to calculate τ_1 and τ'_1. In this fashion, the iteration scheme proceeds forward until τ_N and τ'_N values are obtained. Accuracy of the calculated τ_i values is checked by using the equilibrium condition and an error value is calculated.

Iteration proceeds with new guess values for τ'_0 until the calculated error is less

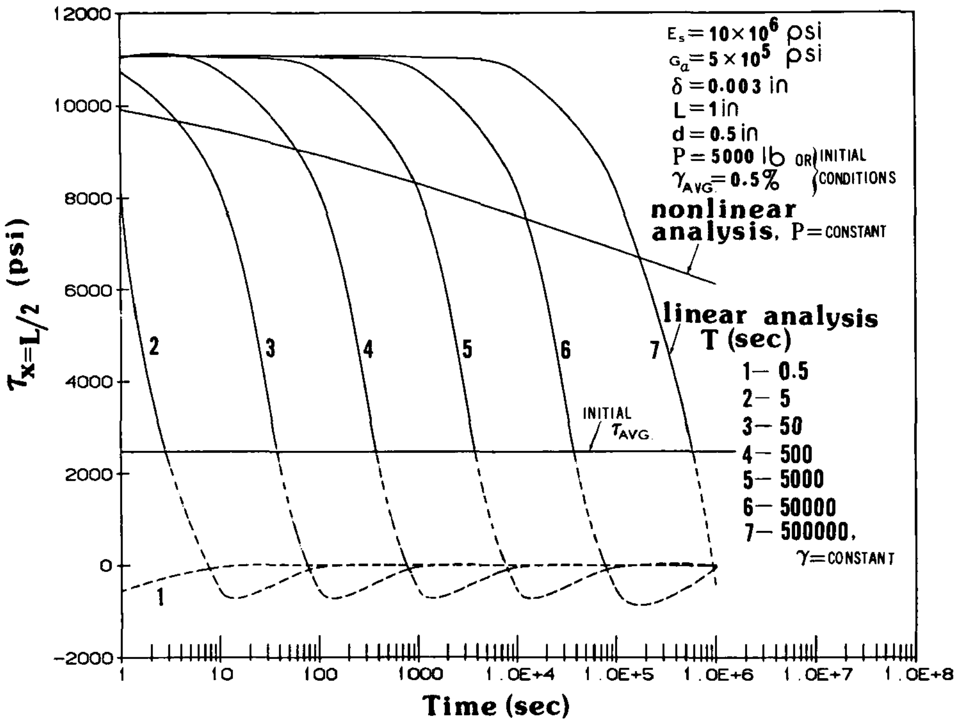

FIGURE 3 Variation of Adhesive Shear Stress at the Overlap Edges of Double Lap Joints as Predicted by Linear and Nonlinear Viscoelastic Analyses.

than 10^{-8}. The choice of guess values for $[\tau'_0]_j$ is determined using an optimal selection method which can be summarized as follows. Initially, based on early numerical trials, the range of guess values is chosen to be between 10^{-60} to 2×10^3 and the midrange value of 1×10^3 is used as the first guess. If the resulting error value is positive a new $[\tau'_0]_{j+1}$ value equal to 5×10^2 and corresponding to midrange between 10^{60} and 1×10^3 is tried. If the initial error is negative, however, $[\tau'_0]_{j+1}=1.5\times10^3$ is used as the next guess. This narrowing selection of midrange values continues until the accuracy criterion error$\leq10^{-8}$ is met.

RESULTS AND CONCLUSIONS

By using typical material properties mentioned earlier for the linear elastic substrate (aluminum) and the viscoelastic adhesive (epoxy), Equations (19) and (20) can be solved as described above to yield the maximum adhesive shear stress values as functions of time. Figure 3 shows comparison of these results. It can be seen that if the correspondence principle is applied, the use of the Maxwell chain is necessary to approximate the continuous change in the relaxation time "T" and to coincide with the results calculated using the nonlinear viscoelastic theory. For this purpose numerical iteration can be performed to determine the relaxation time values which will allow matching the results of Equation (19) with those of Equation (20). Results of such analysis is shown in Figure 4. The time dependent equation for the relaxation time fitted to match the results of nonlinear viscoelastic analysis is obtained as

$$T(\text{sec}) = 9.7t^{0.9055} + 4.32t^{0.6100}. \tag{23}$$

Note that the numerical optimization for Figure 4 was performed using time values equal to or larger than 1 second. Based on Equation (23) a relaxation time value of $T = 0.4955$ sec is predicted for very short duration of $t = 0.013$ sec. Examination of Figure 3 reveals that for this relaxation time value the closed form equation (19) predicts invalid stress levels at short times (*i.e.* less than 10 sec). This is a limitation of Equation (19) due to the use of Heaviside step function in representing the relaxation condition and the curve corresponding to $T = 0.5$ sec has been included in Figure 4 to illustrate this limitation. This limitation is asymptotic in nature and is restricted to very short relaxation time values (*i.e.* $T<0.5$ sec). It should be noted that this limitation of the closed form solution does not affect its accurate application to predict variable relaxation time values (T) with higher magnitudes and as func-

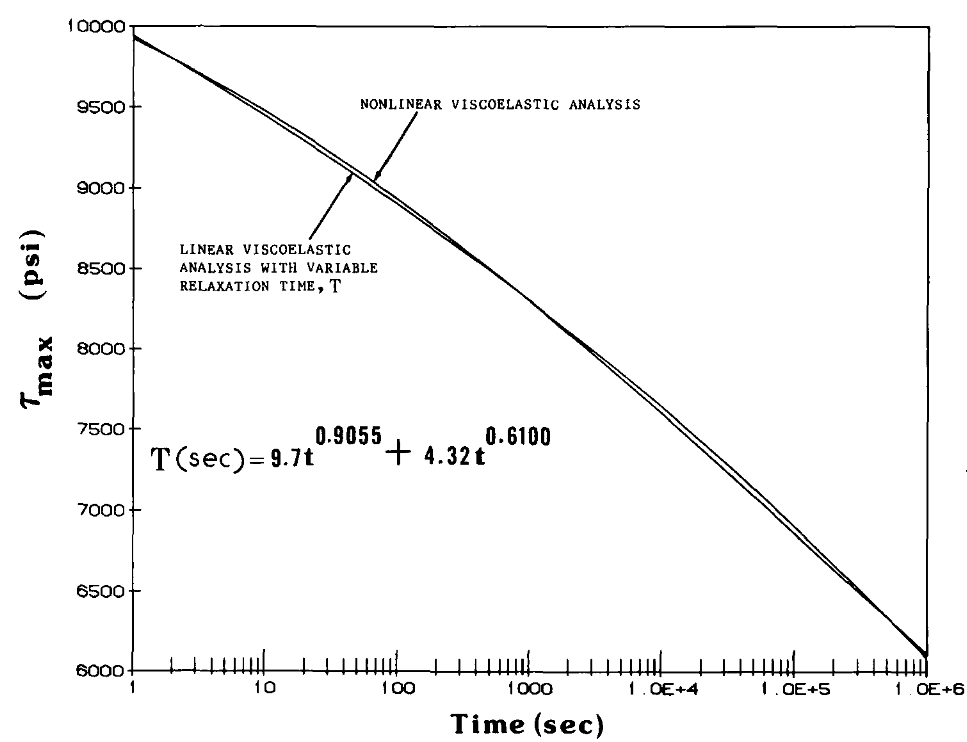

FIGURE 4 Variation of Maximum Adhesive Shear Stress in Double Lap Joints Based on Nonlinear Viscoelastic Analysis and Linear Viscoelastic Analysis with Variable Relaxation Time.

tions of time (t) to obtain a functional form in consort with the nonlinear solution. Prediction difficulties of this sort for very short durations exist for numerical solutions also and are usually the reason for use of time values equal or larger than one second.

In conclusion, Figure 4 reveals the capability of obtaining accurate predictions with the use of the correspondence principle under fixed displacement conditions when variable relaxation time is used. It is necessary, however, to determine how the relaxation time varies by using experimental results.

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